On the probability description of physical reality

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We proposed the concept of reciprocal wave function which is interpreted as the relative non-finding probability amplitude that represents the physical reality in a probabilistic way, like the wave function does. In this note we explore a further question which arises naturally: Why this novel form of probability description of physical reality could exist in quantum theory.

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I. Introduction

The wave function is a basic concept in wave mechanics [1], and one of the postulates in quantum mechanics is Born's probability interpretation of the wave function [2]. The postulate states that the chance of finding a particle in a configuration space is proportional to the value of the absolute square of its wave function at that point, i.e., the wave function is understood as a probability amplitude which represents the physical reality for the particle in the quantum theory.

In the previous work [3] we introduced a new but related form of wave function, which reads

$$\bar{\psi}(q) = \frac{1}{\psi(q)}, \tag{1}$$

where $\psi(q)$ is the Schrödinger wave function in the configuration space. We noted that $\bar{\psi} = \psi$ in physics: the reciprocal wave function $\bar{\psi}$ could represent the same physical reality as the wave function ψ does, and that it is interpreted as the *relative* non-finding probability amplitude for the physical state. Then there is the representation transform for the reciprocal forms of wave function and the other derivative functions in a consistent scheme, i.e.,

$$\left.\begin{aligned} \langle \mathbf{r}|\bar{\psi}\rangle &= \bar{\psi}(\mathbf{r}), \\ \langle \mathbf{p}|\bar{\psi}\rangle &= \bar{\psi}(\mathbf{p}), \end{aligned}\right\} \tag{2}$$

if $\langle \mathbf{r}|\psi\rangle = \psi(\mathbf{r})$ and $\langle \mathbf{p}|\psi\rangle = \psi(\mathbf{p})$. As an application of the transform for the reciprocal form function [3], we are able to calculate the vertex correction in quantum electrodynamics in the nonperturbative region, and also show the well-known phenomenologically-defined Sommerfeld factor [4] is actually the vertex correction.

Although the physical meaning of the reciprocal wave function has been stated, one might agree that the existence of this probability description in quantum theory is very unexpected. Therefore, due to the curiosity which is irresistible to us, we naturally want to ask ourselves a fundamental and perhaps unanswerable question: Why quantum theory admits the concept of the reciprocal wave function which also represents the physical reality. We shall pursue it in this work. Undoubtedly, making progress on this question will offer us fresh and deep insights into the probabilistic description of our world in quantum theory[1].

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[1] Note that theories of quantum mechanics and quantum field theory both, of course, need their probabilistic descriptions for physical reality.

I shall now underline a simple similarity in arithmetic as the starting point. We know that for numbers, $a - b = -[(-a) + b]$, and using the reciprocal operation $\mathcal{D}$, i.e., $\mathcal{D}b = b^{-1}(b \neq 0)$, we may write $a \div b = (\mathcal{D}^2 a) \times (\mathcal{D}b) = \mathcal{D}[(\mathcal{D}a) \times b]$, where $a, b \neq 0$. One sees that the reciprocal operation plays a similar role in multiplication and division as the minus operation in addition and subtraction.

II. The relative information

We now apply a simple classification to informations. The information may be called absolute information, if it is not a information that reflects a comparison. The information may be called relative information, if it reflects a comparison.

The concept of absolute information is not interesting, and we shall focus on the relative information. Clearly, there are two basic kinds of relative informations in the view of mathematics: one is defined by subtraction; the other is defined by division.

After noticing the similarity between minus and reciprocal operations in arithmetic, for the two categories of relative informations, we are naturally interested to consider the similar roles that the operations of minus and reciprocal might play in the formulations.

In fact we may observe that the mathematical consequence of applying the reciprocal operation to the numerical result from the relevant comparison can be, logically, equivalent to changing the original order of the two objects in the comparison, and the similar fact also occurs for the minus operation. The abstract change for the comparison, of course, has no effect on the reality, and however it does have an effect on our description for the reality. Formally, for the descriptions the minus and the reciprocal operations can turn the original relative informations into the informations in "opposite form". Here the opposite form means the numerical result for a comparison actually comes from the comparison in the opposite order.

Due to the feedback on the early version of this work, we find that it can be very helpful to illustrate the meaning of opposite form by specific examples. Thus we here give an enlightening example.

If we compare the area of A with B in Fig. 1, normally in the order we would say: the area of A is 2 cm^2 larger than B. However, to describe the same reality, we can also say: the area of A is -2 cm^2 smaller than B. And this is an example for the relative information in the opposite form, since the value -2 comes from the comparison in the opposite order.

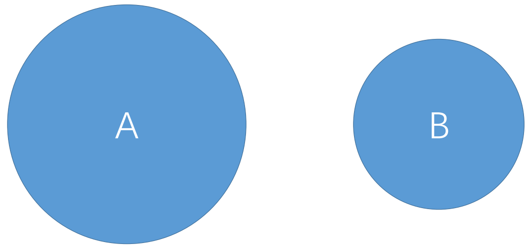

FIG. 1. Two circles A and B. Suppose, for example, that the area of A is 6 cm^2 and that B is 4 cm^2.

Besides areas of surfaces, there are numerous types of quantities of objects in comparisons: lengths, volumes, weights, angles, etc. One may easily find simple examples that can be helpful for understanding the notion of the relative information in the opposite form. It is expected that sometimes the reality described in the opposite form can be awkward for us, and this is also true for the relative information that describes a thing in the "normal form"(changing the comparison order of the circles A and B say). Since the informations in the two forms reflect the same reality, they may be called identical informations.

Next, for our purpose, we shall focus on the concept of the relative probability which is obviously a kind of relative information.

III. The relative probabilities

We know that the Schrödinger wave function is postulated to be the finding probability amplitude for the bound state problem and the *relative* finding probability amplitude for the scattering or free state problem [5].

a. Scattering state

For the continuous spectrum, the finding probability amplitude cannot have a meaning in an absolute sense. Thus, logically the relative probability descriptions of finding and not finding are equal, and it is easily understood that the wave function and the reciprocal wave function describes the same physical reality [3]. Therefore, the concept of reciprocal wave function may actually be understandable for the scattering state.

Nevertheless, we may give a note about the free state. In general, the reciprocal of the wave function $\psi(\mathbf{r})$ (say) would not satisfy the Schrödinger equation for $\psi(\mathbf{r})$. However, the plane wave is kind of exceptive, since the reciprocal and the complex conjugate are the same for $\psi_0(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$ from the mathematical viewpoint. Then the abstract quantity $e^{-i\mathbf{k}\cdot\mathbf{r}}$ can have an independent physical meaning, i.e., we may define that it represents the free particle carrying the momentum $-\mathbf{k}$, and in this case it could be useful to notice that the reciprocal wave function $\bar{\psi}_0(\mathbf{r})$ represents the different physical reality.

Realize that the absolute squares of the wave function and the reciprocal wave function are exchangeable under the reciprocal operation. We may remark that, from a more formal standpoint, the relative non-finding probability description may be viewed as a logical interpretation for the description of the relative finding probability in the opposite form, due to the fact that the finding and the nonfinding are contrary to each other.

b. Bound state

For the wave function of the discrete spectrum, we know that it is square integrable, and after normalization the finding probability can have a meaning in an absolute sense. Consequently, there is a classical non-finding probability description for the bound state, which is apparently not formulated by the reciprocal wave function. Then it seems that the concept of the reciprocal wave function would not make any sense for the bound state problem. But, our point is that the relative non-finding probability is introduced as a universal concept, and that it would not be only valid for the scattering state. Thus, let us now consider this negative-perspective probability description for the bound state.

From the purely practical viewpoint, indeed, the wave function and the reciprocal wave function would not be equal, since the relative non-finding probability cannot have an absolute meaning for seeking the position of a particle in a configuration space. However, we may recall a basic fact in quantum mechanics: the relative finding probability is more important than the absolute finding probability for our abstract description of the physical reality. This fact is easily appreciated, since ψ and $c\psi$ represent the same physical reality, where c is any, not zero, constant. And notice that the procedure of normalization is only relevant to the concept of the probability-related measurement. Thus, from the point of view of the abstract description of the physical reality, the situation of the bound state is not essentially different from the scattering state.

However, to make our point more clear, we shall show a failure for the concept of the absolute non-finding probability in quantum theory.

Let a normalized wave function in the coordinate representation $\psi(\mathbf{r})$ represents a state of a bounded particle. The classical non-finding probability $P_{\text{non}}(\mathbf{r})$ for a "point", an element of volume, is[2]

$$P_{\rm non}(\mathbf{r}) = 1 - |\psi(\mathbf{r})|^2 d^3 r. \tag{3}$$

It may be remarked here that this classical non-finding probability is not an information merely regarding that point: it must be equal to the total probability of finding the particle elsewhere.

The probability of detecting the bounded particle in the volume element d^3r, generally, would not be a finite value. Thus, the ratio of the classical non-finding probabilities at two points

$$R_{\rm non}(\mathbf{r_1}, \mathbf{r_2}) = \frac{P_{\rm non}(\mathbf{r_1})}{P_{\rm non}(\mathbf{r_2})} = 1. \tag{4}$$

This formula tells us that the possibilities of not finding the particle at different points are the same. One cannot deny this result is perfectly correct, especially for considering the real points in the configuration space instead of the elements of volume d^3r, but we should also admit this formulation cannot adequately characterize the situation from the viewpoint of nonfinding. Besides, it loses entirely the relative information formulated by the finding probability amplitude at two points, i.e., the probability ratio $R[= |\psi(\mathbf{r}_i)|^2/|\psi(\mathbf{r}_j)|^2]$, which is the information that really matters for the probability description of the physical reality. That is to say, for this classical non-finding probability ratio, when you are informed about the correct information for every two points in the configuration space, you still cannot determine the probability description of a state.

Although it may not be the root cause for this failure, it is instructive to notice the fact that for a comparison the relative informations separately defined by subtraction and division are not identical informations and that the classically-compared formulation mixed subtraction and division.

For the relative non-finding amplitude, we may understand that it is introduced to define a type of information that can simply concern a point from the non-finding perspective. Hence, there is a simple equivalent relation that $\bar{R}(\mathbf{r_1}, \mathbf{r_2}) = \bar{R}(\mathbf{r_3}, \mathbf{r_4})$, if $R(\mathbf{r_1}, \mathbf{r_2}) = R(\mathbf{r_3}, \mathbf{r_4})$, and conversely, where $\bar{R}[= |\bar{\psi}(\mathbf{r}_i)|^2/|\bar{\psi}(\mathbf{r}_j)|^2]$ is the probability ratio formulated by the relative non-finding probability amplitude.

Perhaps a note here can be helpful. The bound state wave function goes to zero at infinity, and then the reciprocal wave function would be divergent at infinity. However, we should always keep in mind that the relative non-finding probability amplitude would not be meaningful in an absolute sense. Thus it is not necessary to worry about possible divergences at some points. As a

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[2] We may emphasize again that this classical non-finding probability formula does not applicable for the continuous spectrum, since we cannot have the total probability "1".

matter of fact, the value of the relative non-finding probability at a point or infinitesimal interval, $1/(|\psi|^2 dq)$, almost always diverges, and that does not matter.

IV. Remarks

The notion of the relative non-finding probability is not derived from classical concepts. Thus at first sight it might look odd to us. However, we have argued that it is a logical result after careful consideration, i.e., the existence of this novel probability description of physical reality is inherent in quantum theory. We hope this provides fresh and nontrivial insights into the probabilistic description of physical reality.

Acknowledgments

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[5] See, for example, P. A. M. Dirac, *The principles of quantum mechanics*, Oxford university press, 1958, p. 73.